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Géza Freud's Work on Tauberian Remainder Theorems

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DEDICATED TO THE MEMORY OF GÉZA FREUD

1. TAUBERIAN THEOREMS

The following theorem due to Abel should be well known by every student of mathematics:

If $(a_n)_0^\infty$ is a real sequence with

$$\sum_0^\infty a_n = s, \tag{1.1}$$

then

$$\lim_{x \uparrow 1} \sum_0^\infty a_n x^n = s. \tag{1.2}$$

The converse result, that (1.2) implies (1.1), is not true without further assumptions as the example

$$\sum_0^\infty a_n x^n = \tfrac{1}{2}(1+x)^{-1}(1-x)$$

shows, where $a_0 = \frac{1}{2}$, $a_n = (-1)^n$ for $n = 1, 2, \ldots$. The first converse theorem was given by A. Tauber in 1897, who proved that

$$na_n = o(1)$$

is a condition that together with (1.2) implies (1.1). The theory was further developed by Hardy and Littlewood, who named such converse theorems "Tauberian" and gave applications, for instance, to number theory. The

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most famous of their Tauberian theorems (Hardy and Littlewood [10]) reads:

If $na_n \geqslant -1$ *for all* n, *then* (1.2) *implies* (1.1).

A closely related result says that, *if* α *is positive, then*

$$\lim_{x \uparrow 1} (1-x)^\alpha \sum_0^\infty a_n x^n = s \tag{1.3}$$

and

$$n^{1-\alpha} a_n \geqslant -1, \tag{1.4}$$

imply that

$$\lim N^{-\alpha} \sum_0^N a_n = \frac{s}{\Gamma(1+\alpha)}. \tag{1.5}$$

It is easy to check that no generality is lost, if we replace (1.4) by $a_n \geqslant 0$. In this form the theorem is a good starting point for the introduction of Tauberian remainder theorems and the presentation of Freud's fundamental result in this field.

2. Tauberian Remainder Theorems for the Laplace Transform

In a remainder theorem we assume that we have some information on the speed of convergence in (1.3), for instance, that

$$(1-x)^\alpha \sum_0^\infty a_n x^n = s + o((1-x)^\delta), \qquad \delta > 0.$$

Does that give us more and better information than (1.5), if $a_n \geqslant 0$? The answer is yes!

It is a little easier to write the formulas, if we replace the power series by the more general Laplace transform. The Tauberian theorem given at the end of Section 1 can be restated:

If τ *is a nondecreasing function on* $[0, \infty)$ *and* α *is positive, then it follows from*

$$\lim_{t \downarrow 0} t^\alpha \int_0^\infty e^{-tu}\, d\tau(u) = s,$$

that

$$\lim_{x \to \infty} x^{-\alpha} \tau(x) = \frac{s}{\Gamma(1+\alpha)}.$$

The first precise remainder estimate in this theorem was given by Freud in [1]. Several mathematicians were working on the problem at this time. Somewhat weaker results were published the same year by Korevaar and Postnikov. The main contributions besides Freud's were given by Korevaar in a sequence of papers (Korevaar [13–15]). Among other things Korevaar had provided an example showing that Freud's seemingly weak estimate cannot be improved. Freud once told me that he had had his estimate for some time but first when he heard about Korevaar's example he dared to publish it.

I next state Freud's main theorem. We shall discuss some of the other results he obtained in his papers (Freud [1, 2]) in Section 5.

FREUD'S THEOREM. *Let τ be nondecreasing on $[0, \infty)$ and let α and δ be positive numbers. If*

$$F(t) = \int_0^\infty e^{-tu}\, d\tau(u) = st^{-\alpha} + o(t^{\delta-\alpha}), \qquad t \downarrow 0, \tag{2.1}$$

then

$$\tau(x) = \frac{sx^\alpha}{\Gamma(1+\alpha)} + O\left(\frac{x^\alpha}{\log x}\right), \qquad x \to \infty. \tag{2.2}$$

The crucial lemma used in the proof concerns one-sided approximation in L of functions of bounded variation by polynomials. That a result on the degree of polynomial approximation is fundamental, is no surprise for those who know the beautiful method Karamata invented for the proof of Tauberian theorems (Karamata [11], Wielandt [18]). Both Freud and Korevaar applied the Karamata method. Before going into a closer study of Freud's work, I shall give a short review of known results on Tauberian remainders at the time.

3. TAUBERIAN REMAINDER THEOREMS IN GENERAL

In my mathematical surroundings in Stockholm in the late 1940s there was an air of mysticism around the topic "remainders in Tauberian theorems." It emanated of course from a statement in Wiener's famous paper (Wiener [19]). There is a sentence that evidently can be interpreted in different ways. My guess is that it just means that you cannot say anything about the remainder in Wiener's general Tauberian theorem without further assumptions about the kernel. In fact, in 1938 Beurling had already given a precise remainder estimate for a special class of kernels. However, the proofs were not published until 1954 in Lyttkens' thesis

(Lyttkens [16]). To see how that result and other later Tauberian remainder theorem's in Wiener's form are related to Freud's, we shall transform Freud's theorem as stated in Section 2 into Wiener's form. The computations are rather tedious, but the main part goes as follows: Introducing ϕ defined by

$$\phi(y) = e^{-\alpha y}\tau(e^y) - \frac{s}{\Gamma(1+\alpha)}$$

instead of τ, assuming that $\tau(0)=0$ and putting $t=e^{-x}$ and $u=e^y$, we obtain after an integration by parts in (2.1) that

$$\int_{-\infty}^{\infty} \exp(-(1+\alpha)(x-y) - \exp(y-x))\,\phi(y)\,dy = O(e^{-\delta x}).$$

Some checking of magnitudes is necessary, but the result is, that formula (2.1) can be written in Wiener's form

$$K * \phi(x) = O(e^{-\delta x}), \qquad x \to \infty, \tag{3.1}$$

where $K * \phi(x) = \int_{-\infty}^{\infty} K(x-y)\ \phi(y)\ dy$ and ϕ defined above belongs to $L^\infty(-\infty, \infty)$ and has the property that for some constant C it holds that

$$\phi(x) + Cx \qquad \text{is nondecreasing.} \tag{3.2}$$

The kernel K is given by

$$K(x) = \exp(-(1+\alpha)\,x - \exp(-x)), \tag{3.3}$$

and evidently $K \in L(-\infty, \infty)$ with $\hat{K}(t) = \int_{-\infty}^{\infty} e^{-itx}K(x)\ dx = \Gamma(1+\alpha+it)$. The conclusion (2.2) takes the form

$$\phi(x) = O(1/x), \qquad x \to \infty.$$

It can be shown that Freud's theorem of Section 2 is equivalent to the result that, for the special choice of kernel (3.3), the estimate given in (3.1) combined with the Tauberian condition (3.2) implies that $\phi(x) = O(1/x)$. At this point it is natural to ask, if the result just mentioned is a special case of a general theorem. The only general remainder known in 1951 was Beurling's, and that concerned kernels for which $\hat{K}(t)^{-1}$ does not grow more rapidly than a polynomial of degree n in a strip around the real axis, the conclusion being that (3.5),

$$\phi(x) = O\left(\exp\left(-\frac{x}{n+1}\right)\right).$$

Now it is well known that for the kernel corresponding to the Laplace-transform $\hat{K}(t)^{-1} = \Gamma(1+\alpha+it)^{-1}$ is of exponential growth, hence not in the class mentioned above. Accordingly the best possible estimate (3.4) is much weaker than (3.5). On the other hand Beurling's theorem is applicable to a large class of kernels including those corresponding to Cesàro summation. The first general remainder theorem covering Freud's was not proved until 1962, when I obtained the conclusion (3.4) from the assumptions (3.1) and (3.2) for kernels in a class E defined as follows.

An integrable function K belongs to E, if there is a function g holomorphic in a strip $|\text{Im } z| < b$, such that

$$|g(z)| \leqslant M \exp \text{Im } z) \qquad \text{for} \quad |\text{Im } z| < b$$

and

$$g(x)\,\hat{K}(x) = 1 \qquad \text{for all real } x.$$

Since then many general remainder theorems have been obtained, applicable to various special kernels (cf. Ganelius 1972). There are of course also many new special Tauberian remainder theorems. However, Freud was the first to obtain the precise result in the important case of the Laplace transform.

4. Freud's Approximation Theorem and the Proof of the Tauberian Remainder Theorem

As remarked in Section 2 the crucial point in a method to obtain sharp remainder estimates is the right approximation theorem. I shall now state Freud's approximation theorem without reproducing the proof. It should be mentioned, that Freud's deep knowledge of approximation theory enabled him to obtain the following result by a rather straight forward application of a well known construction by Markov (cf. also Freud [2] and b for $\alpha \geqslant \frac{1}{2}$ and Nevai [17] for $\alpha > 0$).

Freud's Approximation Theorem. *If g is a function of bounded variation on* [0, 1] *and α is positive, then polynomials p and P of degree n can be found, such that*

$$p(x) \leqslant g(x) \leqslant P(x), \qquad 0 \leqslant x \leqslant 1,$$

and

$$\int_0^1 (P(x) - P(x)) \left(\log \frac{1}{x}\right)^{\alpha - 1} dx \leqslant cn^{-1}.$$

Moreover, if $p(x)=\sum b_k x^k$, $P(x)=\sum B_k x^k$, *then there are constants* d *and* f *independent of* n *and such that* $\sum_0^n (|b_k|+|B_k|)\leqslant d\exp(fn)$.

The approximation theorem will only be applied to the case

$$g(x)=\begin{cases}0 & \text{if } 0\leqslant x<1/e \\ x^{-1} & \text{if } 1/e\leqslant x<1.\end{cases} \tag{4.1}$$

We shall now see how easy the remainder theorem follows from the approximation theorem if we apply Karamata's method. Introducing $\sigma(u)=\tau(u)-(s/\Gamma(1+\alpha))\,u^\alpha$ we rewrite the assumption (2.1) in Freud's theorem

$$r(t)=\int_0^\infty e^{-tu}\,d\sigma(u)=O(t^{\delta-\alpha}). \tag{4.2}$$

The conclusion shall be

$$\sigma(x)=O\left(\frac{x^\alpha}{\log x}\right),\qquad x\to\infty. \tag{4.3}$$

With g given by (4.1) and $x=t^{-1}$ we obtain

$$\begin{aligned}\sigma(x)&=\int_0^\infty g(e^{-tu})\,e^{-tu}\,d\sigma(u)\\ &=\int_0^\infty (g(e^{-tu})-p(e^{-tu}))\,e^{-tu}\,d\sigma(u)+\int_0^\infty p(e^{-tu})\,e^{-tu}\,d\sigma(u)\\ &\geqslant -\frac{s}{\Gamma(1+\alpha)}\int_0^\infty (g(e^{-tu})-p(e^{-tu}))\,e^{-tu}\,du^\alpha-\sum_0^n |b_k|\,|r((k+1)\,t)|\\ &\geqslant -\frac{\alpha s x^\alpha}{\Gamma(1+\alpha)}\int_0^1 (g(v)-p(v))\left(\log\frac{1}{v}\right)^{\alpha-1} dv-x^{\alpha-\delta}\sum_0^n |b_k|(k+1)^{\delta+\alpha}\\ &\geqslant -c(n^{-1}x^\alpha+e^{Cn}x^{\alpha-\delta}).\end{aligned}$$

In this computation we have used $v=e^{-tu}$ as new variable. Taking $n=\varepsilon\log x$ with a sufficiently small ε, we obtain the lower bound in (4.3). Repeating the computation with the polynomial P instead of p we get the upper bound and the theorem is proved.

5. Extensions and Generalizations

The remainder theorem of Section 2 is the principal result in Freud's paper published in 1951. Already in that paper Freud points out that his method works for more general remainders than the $O(t^{\delta-\alpha})$ in (2.1).

Let τ be nondecreasing and let α be a positive number. Assume further that R is nondecreasing with $R(0)=0$ and $R(ks)\leqslant e^{dk}R(s)$. Then

$$\int_0^\infty e^{-tu}\, d\tau(u)=s\Gamma(1+\alpha)\, t^{-\alpha}(1+r(t)), \quad t\downarrow 0,\ |r(t)|\leqslant R(t),$$

implies that

$$\tau(x)=sx^\alpha(1+\rho(x)), \qquad x\to\infty,$$

where

$$|\rho(x)|\leqslant\frac{c}{\log R(1/x)}.$$

The proof follows by the same method. The condition on the function R excludes rapidly decreasing remainders like $\exp(-ct^{-\beta})$. It is of a certain interest that Freud's method is good enough even in such cases, which in fact are important for applications to the spectral theory of differential operators.

That Géza Freud left out such cases in order to get a more lucid theorem, gave me the opportunity to show (Ganelius [6, 7]) that a straightforward application of his method gives sharp results of interest in the field of differential equations. A simple result in this direction is as follows.

Let τ and α be as in the previous theorem and assume that

$$\int_0^\infty e^{-tu}\, d\tau(u)=s\Gamma(1+\alpha)\, t^{-\alpha}(1+r(t)), \qquad t\downarrow 0$$

with

$$|r(t)|\leqslant c_0\exp(-ct^{-\delta}), \qquad 0<\varepsilon\leqslant 1.$$

Then

$$\sigma(x)=\tau(x)-sx^\alpha=O(x^{\alpha-\varepsilon/(1+\varepsilon)}), \qquad x\to\infty.$$

In the applications of this result it is important that smaller remainders are obtained for the Cesàro means of the function, viz.

$$\int_0^x (1-v/x)^{p-1}\, d\sigma(v)=O(x^{\alpha-p\varepsilon/(1+\varepsilon)}), \qquad x\to\infty.$$

(For applications of such results to summability problems for eigenfunction expansions cf. Bergendal [4].)

Generalizations of the type just mentioned and methods to obtain them were given in Freud's second paper. The pattern is as simple as in the example above:

If $\sigma(x) = O(x^\alpha/\log R(1/x))$, *then* $\int_0^x (1 - v/x)^{p-1}\, d\sigma(x) = O(x^\alpha/(\log R(1/x))^p)$.

In this third paper Freud gave some further extensions and applications. The result I find most interesting has the following concise form in the case of power series:

If, with $\delta > 0$, *it holds that*

$$\sum_0^\infty a_n e^{-ns} = A + O(s^\delta), \qquad s \downarrow 0,$$

and

$$\lim_{n\to\infty} \inf \frac{na_n}{\log n} \geqslant 0,$$

then $\sum a_n$ *converges and has the sum* A.

This means that under the weaker tauberian condition we still get convergence. The beauty of the result is reinforced by another counter example of Korevaar's showing that the Tauberian condition $a_n = O(\log n/n)$ is not sufficient for convergence.

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